

# Asymptotic properties of the quantum representations of the mapping class group

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## Abstract

We establish various results on the large level limit of projective quantum representations of surface mapping class groups obtained by quantizing moduli spaces of flat  $SU(n)$ -bundle. Working with the metaplectic correction, we proved that these projective representations lift to asymptotic representations. We show that the operators in these representations are Fourier integral operators and determine explicitly their canonical relations and symbols. We deduce from these facts the Egorov property and the asymptotic unitarity, two results already proved by J.E. Andersen. Furthermore we show under a transversality assumption that the characters of these representations have an asymptotic expansion. The leading order term of this expansion agrees with the formula derived heuristically by E. Witten in *Quantum field theory and the Jones polynomial*.

Quantum Chern-Simons theory was introduced twenty years ago by Witten [32] and Reshetikhin-Turaev [30]. It provides finite dimensional projective representations of surface mapping class groups. These representations may be defined with combinatorial-topological methods or through conformal field theory. Another approach, that we will follow, consists in quantizing the moduli spaces of flat principal bundles with a given structure group.

More precisely, consider a closed surface  $\Sigma$ , a compact Lie group  $G$  and an integer  $k$ , called the level. The moduli space of flat  $G$ -principal bundles over  $\Sigma$  is the base of a Hermitian line bundle called the Chern-Simons bundle. The quantum Hilbert space associated to  $(\Sigma, G, k)$  is the space of

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holomorphic sections of the  $k$ -th tensor power of the Chern-Simons bundle. Here the complex structure is induced by a class of complex structures in the Teichmüller space of  $\Sigma$ . So we have a vector bundle over the Teichmüller space, called the Verlinde bundle, whose fibers are the various Hilbert spaces. Hitchin [22] and Axelrod-Della Pietra-Witten [6] proved that the Verlinde bundle has a natural projectively flat connection. Using this connection, we can identify the different fibers through parallel transport. The connection being equivariant with respect to the mapping class group action, we obtain a projective representation of this group.

In this paper, we study the asymptotic properties of the connection and deduce several facts on the mapping class group representations. Here and in the sequel, asymptotic always refer to the large level limit.

We consider only the case where  $G$  is the special unitary group  $SU(n)$  with  $n \geq 2$ , the genus  $g$  of  $\Sigma$  is larger than 2 and the moduli space is the space of  $G$ -principal flat bundles whose holonomy around a given marked point is a fixed generator of the center of  $G$ . With this last condition the moduli space is smooth. The relevant mapping class group here is  $\Gamma_g^1$ , the group of isotopy classes of orientation preserving homeomorphisms that are the identity on a fixed neighborhood of the marked point. Using the connection of the Verlinde bundle defined in [22], we obtain a projective representation of  $\Gamma_g^1$ .

We work with the metaplectic correction. This does not change the Verlinde bundle, neither the projective representation. Nevertheless the connection is modified by a multiple of identity and its natural automorphisms group is an extension of the mapping class group

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{\Gamma}_g^1 \rightarrow \Gamma_g^1 \rightarrow 1.$$

This modified connection has been introduced in [4], the comparison with the one of [22] is made in the present paper.

Our first semi-classical result says that the modified connection is asymptotically flat in the sense that its curvature is bounded on compact subsets by a constant times the inverse of the level, cf. theorem 2.1. As a consequence we obtain an asymptotic representation of  $\tilde{\Gamma}_g^1$ . More precisely we can lift the projective representation in such a way that the projective multiplier of any two elements of  $\tilde{\Gamma}_g^1$  is bounded by the inverse of the level.

Second we prove that the operators in these asymptotic representations are Fourier integral operators, the underlying canonical relations being given by the action of the mapping class group on the Chern-Simons bundle and the principal symbols by the action on the half-form bundle, cf. theorems

2.2 and 2.3. So we recover the classical Chern-Simons theory from large level quantum theory. This is our main result and it has several consequences.

First we recover some results proved by Andersen in [2]: the quantum representations satisfy the Egorov property. This implies that an element of  $\tilde{\Gamma}_1^g$  can not act trivially unless its action on the moduli space is trivial, cf. theorem 2.4. Furthermore the quantum representations are asymptotically unitary for a natural scalar product, meaning that the product of each operator by its adjoint is the identity modulo a term bounded by the inverse of the level, cf. theorem 2.5. The existence of a scalar product asymptotically invariant was already proved in [2].

Another important consequence, which is a new result, is the estimate under some transversality assumptions of the characters of the mapping class group representation. We prove that these characters admit an asymptotic expansion and compute explicitly the leading order term, cf. theorem 2.6. It is a general property of topological quantum field theory that the character of the representation of an element  $\varphi$  of the mapping class group is the three-dimensional invariants of the mapping torus of  $\varphi$ . Using path integral methods, Witten derived heuristically [32] an asymptotically equivalent of the three-dimensional invariants. The leading order term we find agrees with Witten's formula. Let us point out that to our knowledge the equivalence between the different approaches of topological quantum field theory remains conjectural. In particular we do not know any precise statement relating the character of the representation defined via geometric quantization of the moduli spaces with the invariants of surface bundles defined via combinatorial-topological methods.

Our proofs rely on the microlocal techniques which have been developed in the context of geometric quantization since the seminal work of Boutet de Monvel and Guillemin [10]. We essentially use results of our own papers: [13] for the Fourier integral operators in this geometric context, [15] for the semi-classical properties of connections and [18] for the estimate of the trace of a Fourier integral operator. The role played by the metaplectic correction in this theory was understood in [15]. For a part of the results, this correction is not necessary. As instance we can prove that the operator in the representation defined with the connection of [22] are Fourier integral operators and that their characters admit an asymptotic expansion. But for the computations of the principal symbol of these operators and of the leading order term of the asymptotic expansion of the characters, we use the metaplectic correction.

The semi-classical limit of the quantum Chern-Simons theory is essentially conjectural, because it relies on the path integral formalism. Neverthe-

less some rigorous results have been obtained, cf. [25] for the estimate of the trace norm of the curve operators, [20] and [2] for the asymptotic faithfulness. In particular, the paper [2] was the first using microlocal techniques in this context, cf also [3]. Many papers were devoted to the Witten's formulas for the asymptotic of the three-dimensional invariant of Seifert manifolds, cf. [7] for a recent account and references therein. The case of the mapping torus of finite order diffeomorphisms has been considered in [1].

In a companion paper [16], we prove similar results in genus 1 for the quantum representations of the modular group. In particular we derive the asymptotic for the the torus bundle invariants proved in [23].

The paper is organized as follows. Section 1 is mainly expository. We introduce the various data necessary to quantize the moduli spaces which are the complex structures, the Chern-Simons bundle and half-form bundle. We also discuss carefully the action of the modular group. Since we work with one marked point, it is necessary to consider various extensions of the usual mapping class group. In section 2, we state the results previously mentioned in the introduction. In section 3 we review the construction of the Hitchin's connection. We give a general presentation in order to compare the connection of [22] with the one of [4]. This makes clear that the connection with the metaplectic correction is projectively flat. In sections 4 and 5, we prove the asymptotic flatness. The material in these sections is completely elementary. In the last sections 6 and 7, we introduce a general class of semi-classical connections and prove that their parallel transport are Fourier integral operators.

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## 1 Quantization of moduli spaces

### 1.1 Moduli space and the Chern-Simons bundle

Let  $\Sigma$  be a compact connected oriented surface without boundary of genus  $g \geq 2$  with a marked point  $p$ . Let  $n, d$  be two coprime integers with  $n \geq 2$ . Consider the moduli space  $\mathcal{M}$  of flat  $\mathrm{SU}(n)$ -principal bundles over  $\Sigma \setminus \{p\}$  with holonomy around  $p$  equal to  $\exp(2i\pi d/n) \mathrm{id}$ . Since  $(n, d) = 1$ ,  $\mathcal{M}$  is a smooth compact manifold.

For any  $[P] \in \mathcal{M}$ , the bundle associated to  $P$  via the adjoint representation is a flat real vector bundle over  $\Sigma \setminus \{p\}$  whose holonomy around  $p$

is trivial. So this associated bundle is the restriction of a flat real vector bundle  $\text{Ad } P$  over  $\Sigma$ , unique up to isomorphism. The tangent space of the moduli space at  $[P]$  is

$$T_{[P]}\mathcal{M} \simeq H^1(\Sigma, \text{Ad } P).$$

The bundle  $\text{Ad } P$  has a natural metric coming from the basic scalar product of  $\mathfrak{su}(n)$

$$a \cdot b = -\frac{1}{4\pi^2} \text{tr}(ab), \quad a, b \in \mathfrak{su}(n)$$

Atiyah and Bott [5] introduced a symplectic form  $\omega_{\mathcal{M}}$  on  $\mathcal{M}$ . It is given by

$$\omega_{\mathcal{M}}([a], [b]) = 2\pi \int_{\Sigma} a \cdot b$$

where  $a$  and  $b$  are any closed forms of  $\Omega^1(\Sigma, \text{Ad } P)$ .

The following facts are proved in [5], [29], [27]:  $\mathcal{M}$  is simply connected, it has no torsion, its second Betti number is one and  $n\omega_{\mathcal{M}}$  is a generator of  $H^2(\mathcal{M}, \mathbb{Z}) \subset H^2(\mathcal{M}, \mathbb{R})$ . So there exists a Hermitian line bundle

$$L_{CS} \rightarrow \mathcal{M}$$

equipped with a connection of curvature  $\frac{n}{i}\omega_{\mathcal{M}}$ . We call  $L_{CS}$  the Chern-Simons bundle, it is unique up to isomorphism.

Various explicit constructions of  $L_{CS}$  as a quotient in gauge theory were given in [19], [26] and [9] extending the construction of Ramadas-Singer-Weitsman [28] in the case  $\Sigma$  has no marked point. This is important for our purpose because one can deduce that the mapping class group action on  $\mathcal{M}$  lifts to  $L_{CS}$ , cf. next section. We can also construct by gauge theory an orbifold prequantum bundle on  $\mathcal{M}$  with curvature  $\frac{1}{i}\omega_{\mathcal{M}}$ . The  $n$ th power of this bundle is  $L_{CS}$ . There is no restriction to consider only  $L_{CS}$  because only the  $k$ -th powers of the orbifold prequantum bundle with  $k$  divisible by  $n$  admits non trivial global sections.

## 1.2 Mapping class groups

Since we consider a surface with a marked point  $p$ , we may introduce three different mapping class groups

$$\Gamma_g := \pi_0(\text{Diff}^+(\Sigma)), \quad \Gamma_{g,1} := \pi_0(\text{Diff}^+(\Sigma, p)), \quad \Gamma_g^1 := \pi_0(\text{Diff}^+(\Sigma, D))$$

where  $D$  is an embedded disk in  $\Sigma$  whose interior contains  $p$ . Recall that  $\Gamma_{g,1}$  is an extension of  $\Gamma_g$  by the fundamental group of  $\Sigma$ ,

$$1 \rightarrow \pi_1(\Sigma) \rightarrow \Gamma_{g,1} \rightarrow \Gamma_g \rightarrow 1.$$

Furthermore,

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma_g^1 \rightarrow \Gamma_{g,1} \rightarrow 1$$

where the kernel is generated by a Dehn twist on a loop around  $p$ .

The following facts are explained in detail in the paper [17]. First, the mapping class group  $\Gamma_{g,1}$  acts on  $\mathcal{M}$ ; this action does not factor through an action of  $\Gamma_g$ . Second, the mapping class group  $\Gamma_g^1$  acts the Chern-Simons bundle by automorphisms of prequantum bundles. This action lifts the previous action of  $\Gamma_{g,1}$  on  $\mathcal{M}$ . Nevertheless it does not in general factor through an action of  $\Gamma_{g,1}$ . Indeed, a Dehn twist on a loop around  $p$  acts on  $L_{CS}$  by multiplication by  $\exp(i\pi(n-1)d^2)$  in each fiber.

### 1.3 Complex structure

Suppose  $\Sigma$  is endowed a complex structure compatible with the orientation. Then by Hodge decomposition, for any  $[P] \in \mathcal{M}$

$$H^1(M, \text{Ad } P) \otimes \mathbb{C} = H^{0,1}(\Sigma, (\text{Ad } P) \otimes \mathbb{C}) \oplus H^{1,0}(\Sigma, (\text{Ad } P) \otimes \mathbb{C})$$

$\mathcal{M}$  has a complex structure such that the holomorphic tangent space at  $[P]$  is the first summand in the previous decomposition. This complex structure is integrable, compatible with  $\omega_{\mathcal{M}}$  and positive. So it makes  $\mathcal{M}$  a Kähler manifold. It may also be defined by identifying  $\mathcal{M}$  with the moduli space of holomorphic vector bundles of rank  $n$ , degree  $d$  with a fixed determinant through the Narasimhan-Seshadri theorem. The Chern-Simons bundle has a unique holomorphic structure such that its  $\bar{\partial}$ -operator is the  $(0,1)$ -part of the connection.

Let  $\mathcal{A}$  be the space of complex structures of  $\Sigma$  and

$$\mathcal{T} := \mathcal{A} / \text{Diff}_0^+(\Sigma)$$

be the Teichmüller space. The mapping class group  $\Gamma_g$  acts on  $\mathcal{T}$ . As a fact, the complex structure of  $\mathcal{M}$  induced by  $j \in \mathcal{A}$  only depends on the class of  $j$  in the Teichmüller space. Furthermore the actions of  $\Gamma_{g,1}$  on  $\mathcal{M}$  and  $\Gamma_g$  on  $\mathcal{T}$  are compatible in the sense that for any  $\gamma \in \Gamma_{g,1}$  and  $\sigma \in \mathcal{T}$ , the action of  $\gamma$  on  $\mathcal{M}$  sends the complex structure induced by  $\sigma$  into the one induced by  $\gamma(\sigma)$ .

For any class  $\sigma$  in the Teichmüller space, we let  $\mathcal{M}_\sigma$  be  $\mathcal{M}$  endowed with the complex structure induced by  $\sigma$  and  $L_\sigma \rightarrow \mathcal{M}_\sigma$  be the Chern-Simons bundle with the corresponding holomorphic structure.

We shall consider  $\mathcal{T} \times \mathcal{M} \rightarrow \mathcal{M}$  as a smooth family of complex manifold, so we identify  $\mathcal{M}_\sigma$  with the slice  $\{\sigma\} \times \mathcal{M}$ . Denote by  $L$  the pull-back of  $L_{CS}$  by the projection from  $\mathcal{T} \times \mathcal{M}$  onto  $\mathcal{M}$ . The action of  $\Gamma_g^1$  on  $L$  lifts the diagonal action of  $\Gamma_{g,1}$  on  $\mathcal{T} \times \mathcal{M}$ .

#### 1.4 Half-form bundles

Let  $K$  be the line bundle over  $\mathcal{T} \times \mathcal{M}$  whose restriction to  $\mathcal{M}_\sigma$  is the canonical bundle  $K_\sigma$  of  $\mathcal{M}_\sigma$ . By [29], the canonical class of  $\mathcal{M}$  has a square root. So there exists a line bundle  $\delta$  over  $\mathcal{T} \times \mathcal{M}$  such that  $\delta^2 \simeq K$ . Fix an isomorphism  $\varphi$  from  $\delta^2$  to  $K$ . Since  $\mathcal{T}$  is contractible and  $\mathcal{M}$  simply connected,  $\delta$  and  $\varphi$  are unique up to isomorphism.

The diagonal action of the mapping class group  $\Gamma_g^1$  on  $\mathcal{T} \times \mathcal{M}$  lifts naturally to  $K$ . Denote by  $\gamma_K$  the isomorphism of  $K$  corresponding to the action of  $\gamma \in \Gamma_g^1$ . Then define the subgroup of  $\Gamma_g^1 \times \text{Aut}(\delta)$

$$\tilde{\Gamma}_g^1 = \{(\gamma, \gamma_\delta) / \varphi \circ \gamma_\delta^{\otimes 2} = \gamma_K \circ \varphi\}$$

This group is an extension by  $\mathbb{Z}/2\mathbb{Z}$  of  $\Gamma_g^1$ . This follows again from the fact that  $\mathcal{T}$  is contractible and  $\mathcal{M}$  simply connected. Since  $\Gamma_g^1$  acts on the Chern-Simons bundle, the same holds for its cover  $\tilde{\Gamma}_g^1$ . This is the reason why we consider here  $\Gamma_g^1$  instead of  $\Gamma_{g,1}$ .

For any  $\sigma \in \mathcal{T}$ , we denote by  $\delta_\sigma$  the restriction of  $\delta$  to  $\mathcal{M}_\sigma$ . This line bundle has a holomorphic structure and a Hermitian metric determined by the condition that the restriction of  $\varphi$  is an isomorphism  $\delta_\sigma^2 \rightarrow K_\sigma$  of holomorphic Hermitian bundles.

#### 1.5 Quantization

Define for any positive integer  $k$  and any  $\sigma \in \mathcal{T}$ ,

$$\mathcal{H}_{k,\sigma} := H^0(\mathcal{M}_\sigma, L_\sigma^k \otimes \delta_\sigma)$$

This vector space is finite dimensional and has a natural scalar product defined by integrating the pointwise scalar product of sections against the Liouville measure.

Since the canonical class of  $\mathcal{M}$  is negative, it follows from Riemann-Roch-Hilbert theorem and Kodaira vanishing theorem that the dimension

of  $\mathcal{H}_{k,\sigma}$  does not depend on  $\sigma$ . So by elliptic regularity, the  $\mathcal{H}_{k,\sigma}$ 's are the fibers of a smooth vector bundle  $\mathcal{H}_k$  with base  $\mathcal{T}$ . Furthermore, the actions of  $\tilde{\Gamma}_g^1$  on the Chern-Simons bundle and  $\delta$  induce an action of the same group on  $\mathcal{H}_k$ .

By [4], the bundle  $\mathcal{H}_k$  has a natural  $\tilde{\Gamma}_g^1$ -equivariant connection  $\nabla^{\mathcal{H}_k}$ . It is given by an explicit formula, cf. theorem 3.4. The paper [4] does not address the issue of the projective flatness. Actually this projective flatness follows from the results of [22] as it is explained in section 3.8.

## 1.6 Comparison with the usual construction

By [26], the canonical class of  $\mathcal{M}$  is  $-2c_1(L_{CS})$ . So the fiber bundles  $\delta$  and  $L^{-1}$  are isomorphic. Since the Jacobian variety of each  $\mathcal{M}_\sigma$  is trivial, one can even choose an isomorphism between  $\delta$  and  $L^{-1}$  restricting on each slice  $\mathcal{M}_\sigma$  to an isomorphism of holomorphic vector bundles. So

$$\mathcal{H}_{k,\sigma} \simeq H^0(M_\sigma, L_\sigma^{k-1}) \quad (1)$$

In [22], Hitchin defined a flat connection on the projective space bundle over the Teichmüller space with fibers  $\mathbb{P}(H^0(\mathcal{M}_\sigma, L_\sigma^{k-1}))$ . We prove in section 3.8 that this connection is the same as the one induced by  $\nabla^{\mathcal{H}_k}$  via the isomorphism (1).

The bundles  $\delta$  and  $L^{-1}$  have no reason to be  $\tilde{\Gamma}_g^1$ -equivariant. However, choosing as above an isomorphism compatible with the holomorphic structure in each slice, we obtain an equivariant isomorphism between the projective space bundles. This follows from the trivial fact that the group of automorphisms lifting the identity of a holomorphic line bundle with a connected base is  $\mathbb{C}^*$ . So the projective representation induced by  $\nabla^{\mathcal{H}_k}$  is the same as the one induced by the Hitchin connection. Observe that this projective representation factors through a projective representation of  $\Gamma_{g,1}$ .

## 1.7 A conjecture on $\tilde{\Gamma}_1^g$

Recall that the Hodge line bundle is the complex line bundle over the Teichmüller space whose fiber at  $\sigma$  is  $\wedge^{\text{top}} H^1(\Sigma_\sigma, \mathcal{O})$ . Here  $\Sigma_\sigma$  is the surface  $\Sigma$  endowed with the complex structure  $\sigma$  and  $\mathcal{O}$  is the structural sheaf. Denote by  $\lambda$  the pull-back of the Hodge line bundle by the projection  $\mathcal{M} \times \mathcal{T} \rightarrow \mathcal{T}$ . There is an obvious action of  $\Gamma_g$  on  $\lambda$ . Furthermore the previously defined actions of  $\Gamma_g^1$  on  $K$  and  $L_{CS}^2$  factor through actions of  $\Gamma_{g,1}$ . We conjecture that there exists a  $\Gamma_{g,1}$ -equivariant isomorphism

$$K \simeq L_{CS}^{-2} \otimes \lambda^{n^2-1} \quad (2)$$



A similar result is proved for the moduli space without marked point in [21].

Introduce a symplectic basis of the first homology group of  $\Sigma$ . So the action on the homology defines a morphism  $\Psi : \Gamma_g \rightarrow \mathrm{Sp}(2g, \mathbb{R})$ . Let  $\mathrm{Mp}(2g)$  be the metaplectic group and  $q$  be the projection  $\mathrm{Mp}(2g) \rightarrow \mathrm{Sp}(2g, \mathbb{R})$ . If there exists a  $\Gamma_{g,1}$ -equivariant isomorphism (2), then we have the following group isomorphisms

$$\tilde{\Gamma}_g^1 \simeq \begin{cases} \Gamma_g^1 \times \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \{(\gamma, h) \in \Gamma_g^1 \times \mathrm{Mp}(2g) / \Psi(\gamma) = q(h)\} & \text{if } n \text{ is odd.} \end{cases}$$

## 2 Semi-classical results

### 2.1 Asymptotic flatness

The sequence of connections  $(\nabla^{\mathcal{H}_k})$  is asymptotically flat.

**Theorem 2.1.** *For any vector fields  $X, Y$  of  $\mathcal{T}$ , any compact set  $K$  of  $\mathcal{T}$ , there exists  $C > 0$ , such that the curvature at any  $\sigma \in K$*

$$R^{\mathcal{H}_k}(X, Y)(\sigma) = [\nabla_X^{\mathcal{H}_k}, \nabla_Y^{\mathcal{H}_k}] - \nabla_{[X, Y]}^{\mathcal{H}_k} : \mathcal{H}_{k, \sigma} \rightarrow \mathcal{H}_{k, \sigma}$$

*has a uniform norm bounded by  $Ck^{-1}$ .*

We will propose two different proofs. The first in section 5 is completely elementary. The second one relies on a much more general result, theorem 6.2. In both cases, the result follows from miraculous cancellations, happening only with the metaplectic correction and which were first observed in [15].

Let  $\sigma \in \mathcal{T}$ . Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . For any integer  $k$ , the action of  $h$  restricts to a linear map from  $\mathcal{H}_{k, \sigma}$  to  $\mathcal{H}_{k, \gamma.\sigma}$ . By composing this map with the parallel transport along  $p$ , we obtain an endomorphism of  $\mathcal{H}_{k, \sigma}$  that we denote by  $U_k(h, p)$ . It follows from theorem 2.1 that these maps are defined independently of the choice of  $p$  up to a  $O(k^{-1})$  term, that is for any paths  $p_1$  and  $p_2$  from  $\gamma.\sigma$  to  $\sigma$ ,

$$U_k(h, p_1) = \lambda_k(p_1, p_2) U_k(h, p_2)$$

where  $\lambda_k(p_1, p_2)$  is a sequence of complex numbers equal to  $1 + O(k^{-1})$ . Furthermore we obtain an asymptotic representation of  $\tilde{\Gamma}_g^1$  in the sense that

$$U_k(h, p).U_k(h', p') = \mu_k(p, p') U_k(hh', p''),$$

with  $\mu_k(p, p')$  a sequence of complex numbers equal to  $1 + O(k^{-1})$ .

## 2.2 Semi-classical properties of the quantum representation

Our main result says that the sequences  $(U_k(h, p))_k$  are Fourier integral operators. This means that the Schwartz kernel of such a sequence concentrate in a precise way on a Lagrangian submanifold of  $\mathcal{M} \times \mathcal{M}^-$ . Furthermore the restriction of the Schwartz kernel to this canonical relation is described in first approximation by a flat section of the prequantum bundle  $L_{CS} \boxtimes L_{CS}^{-1}$ . Here, the Lagrangian manifold and the flat section are the graphs of the action of the mapping class  $\gamma \in \Gamma_g^1$  on the moduli space  $\mathcal{M}$  and the Chern-Simons bundle respectively.

To state the result, let us recall our convention for Schwartz kernels. Since  $\mathcal{H}_{k,\sigma}$  is a finite dimensional Hilbert space, we have an isomorphism  $\text{End}(\mathcal{H}_{k,\sigma}) \simeq \mathcal{H}_{k,\sigma} \otimes \overline{\mathcal{H}}_{k,\sigma}$ . The latter space identifies with the space of holomorphic sections of the bundle

$$(L_\sigma^k \otimes \delta_\sigma) \boxtimes (\overline{L}_\sigma^k \otimes \overline{\delta}_\sigma) \rightarrow \mathcal{M}_\sigma \times \overline{\mathcal{M}}_\sigma$$

We call the section associated to an endomorphism its Schwartz kernel.

**Theorem 2.2.** *Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . Then the Schwartz kernel of  $U_k(h, p)$  has the following form*

$$\left(\frac{k}{2\pi}\right)^m F^k(x, y) \otimes f(x, y, k) + O(k^{-\infty})$$

where  $m$  is the complex dimension of  $\mathcal{M}$  and

- $F$  is a section of  $L_\sigma \boxtimes \overline{L}_\sigma$  such that  $|F(x, y)| < 1$  if  $x \neq \gamma.y$ ,

$$F(\gamma.y, y) = (\gamma.u) \otimes \bar{u}$$

for all  $y \in \mathcal{M}$  and  $u \in L_y$  of norm 1, and  $\bar{\partial}F \equiv 0$  modulo a section vanishing to any order along  $\{(\gamma.y, y) / y \in \mathcal{M}\}$ .

- $f(., k)$  is a sequence of sections of  $\delta_\sigma \boxtimes \overline{\delta}_\sigma \rightarrow \mathcal{M}_\sigma \times \overline{\mathcal{M}}_\sigma$  which admits an asymptotic expansion in the  $\mathcal{C}^\infty$  topology of the form

$$f_0 + k^{-1}f_1 + k^{-2}f_2 + \dots$$

whose coefficients satisfy  $\bar{\partial}f_i \equiv 0$  modulo a section vanishing to any order along  $\{(\gamma.y, y) / y \in \mathcal{M}\}$ .

Let  $(F_k \rightarrow N)_k$  be a family of Hermitian bundle over the same base and  $(s_k \in \Gamma(N, F_k))_k$  be a family of section. Then we say that  $s_k$  is  $O(k^{-\infty})$  if for any integer  $N$ , there exists  $C_N > 0$  such that

$$\|s_k(x)\| \leq C_N k^{-N}, \quad \forall x \in N.$$

The previous theorem implies in particular that the sequence of Schwartz kernel of  $U_k(h, p)$  is  $O(k^{-\infty})$  over any compact set of  $\mathcal{M}^2 \setminus \{(\gamma.x, x), x \in \mathcal{M}\}$ . To complete the description of the leading order term, we determine the restriction of  $f_0$  to  $\{(\gamma.y, y) \mid y \in \mathcal{M}\}$ . It depends on the morphism  $\gamma_\delta$  of the half-form bundle.

For any  $x$  in  $\mathcal{M}$  and  $\sigma$  in  $\mathcal{T}$ , let  $E_{\sigma,x}$  be the subspace of  $T_x \mathcal{M}_\sigma \otimes \mathbb{C}$  which consists of the vectors of type  $(1, 0)$ . For any  $\sigma, \sigma' \in \mathcal{T}$ , let  $\pi_{\sigma', \sigma, x}$  be the projection from  $E_{\sigma', x}$  onto  $E_{\sigma, x}$  with kernel  $\bar{E}_{\sigma', x}$ . Recall that by definition of the half form bundle, there is a preferred isomorphism  $\varphi$  from  $\delta^2$  to  $K = \det E^*$ . Since the moduli space and the Teichmüller space are simply-connected, there exists a unique family  $\Psi_{\sigma, \sigma', x} : \delta_{\sigma, x} \rightarrow \delta_{\sigma', x}$  of isomorphisms satisfying

$$\varphi_{\sigma', x} \circ \Psi_{\sigma, \sigma', x}^2 = \pi_{\sigma', \sigma, x}^* \circ \varphi_{\sigma, x},$$

depending continuously on  $\sigma, \sigma'$  and  $x$  and such that  $\Psi_{\sigma, \sigma, x}$  is the identity of  $\delta_{\sigma, x}$  for any  $\sigma$ .

**Theorem 2.3.** *Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . Then for any  $x \in \mathcal{M}$ , the Schwartz kernel of  $U_k(h, p)$  at  $(\gamma.x, x)$  is equal to*

$$\left(\frac{k}{2\pi}\right)^m [(\gamma.u) \otimes \bar{u}]^k \otimes [\Psi_{\gamma.\sigma, \sigma, x}(\gamma_\delta(v)) \otimes \bar{v}] + O(k^{m-1})$$

where  $m$  is the complex dimension of  $\mathcal{M}$  and  $u, v$  are any vectors with norm 1 of  $L_x$  and  $\delta_{\sigma, x}$  respectively.

Theorems 2.2 and 2.3 are consequences of theorem 6.3, the latter being a generalization of theorem 7.1 in [15]. We provide a complete proof in section 7.

### 2.3 Asymptotic faithfulness and asymptotic unitarity

Let us explain the relation with Andersen's work. By the Egorov theorem for Toeplitz operator (theorem 3.3 of [15]), one deduce from theorems 2.2 and 2.3 the following fact: if  $(T_k)_k$  is a Toeplitz operator of  $(\mathcal{H}_{k, \sigma})_k$  with symbol  $f \in \mathcal{C}^\infty(\mathcal{M})$ , then

$$(U_k(\gamma, p))^{-1} T_k U_k(\gamma, p)_k$$

is a Toeplitz operator with principal symbol  $\gamma_{\mathcal{M}}^* f$ . This fact was proved in [2]. Observe that the Egorov property depends only on the projective class of  $U_k(\gamma, p)$  unlike theorems 2.2 and 2.3.

The second part of the following theorem was the main argument to prove the asymptotic faithfulness of the quantum representation in [2].

**Theorem 2.4.** *Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . If  $U_k(\gamma, p) = \text{id}$  when  $k$  is sufficiently large, then  $\gamma$  acts trivially on the Chern-Simons bundle and  $\gamma_\delta = \text{id}$ . If*

$$U_k(\gamma, p)TU_k(\gamma, p)^{-1} = T, \quad \forall T \in \text{End}(\mathcal{H}_{k,\sigma})$$

*when  $k$  is sufficiently large, then  $\gamma$  acts trivially on the moduli space  $\mathcal{M}$ .*

The first part is an immediate consequence of theorems 2.2 and 2.3. The second part follows from Egorov property using the fact that any smooth function of  $\mathcal{M}$  is the symbol of a Toeplitz operator.

Another corollary of theorems 2.2 and 2.3 is the asymptotic unitarity.

**Theorem 2.5.** *Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . Then*

$$U_k(\gamma, p)U_k(\gamma, p)^* = \text{id} + O(k^{-1})$$

*where the  $O(k^{-1})$  is for the uniform norm of operators of  $\mathcal{H}_{k,\sigma}$ .*

This is a general properties of Fourier integral operators whose symbols are half-form morphisms, cf. [15] and [18]. In proposition 2 of [2], Andersen introduced a metric asymptotically preserved by the quantum representations which is likely to be the same as the one in this paper.

## 2.4 Characters of the quantum representation

Let us turn to the asymptotic of the characters of the quantum representations. In [18], we proved that the trace of a Fourier integral operator whose canonical relation intersects transversally the diagonal has an asymptotic expansion. We also computed explicitly the leading order term. By theorems 2.2 and 2.3, these results can be applied to the mapping class group representations.

**Theorem 2.6.** *Let  $h = (\gamma, \gamma_\delta) \in \tilde{\Gamma}_g^1$  and  $p$  be a path of the Teichmüller space from  $\gamma.\sigma$  to  $\sigma$ . Assume that the fixed points of the action of  $\gamma$  on  $\mathcal{M}$*

are all non-degenerate. Then we have

$$\mathrm{Tr}(U_k(\gamma, p)) = \sum_{x \in \mathcal{M}/\gamma, x=x} \frac{i^{m(\gamma_\delta, x)} \cdot u(\gamma, x)^k}{|\det(\mathrm{id} - L(\gamma, x))|^{1/2}} + O(k^{-1})$$

where for any fixed point  $x \in \mathcal{M}$ ,

- $L(\gamma, x)$  is the linear tangent map at  $x$  of the action of  $\gamma$  on  $\mathcal{M}$ ,
- the action of  $\gamma$  on the fiber of the Cherns-Simons bundle at  $x$  is the multiplication by the complex number  $u(\gamma, x)$ ,
- $m(\gamma_\delta, x) \in \mathbb{Z}/4\mathbb{Z}$  is the index of the automorphism  $\Psi_{\gamma, \sigma, \sigma, x} \circ \gamma_\delta$  of  $\delta_{\sigma, x}$ .

The indices  $m(\gamma_\delta, x)$  are defined in [18]. Let us discuss the other terms of the formula. Let  $\Phi$  be a diffeomorphism of  $\Sigma$  fixing  $p$ . Let  $[P] \in \mathcal{M}$  be such that the restriction of  $\Phi$  to  $\Sigma \setminus \{p\}$  lifts to an isomorphism  $\Phi_P$  of  $P$ . Then the induced isomorphism of  $\mathrm{Ad} P \rightarrow \Sigma \setminus \{p\}$  extends by continuity at  $p$  to an isomorphism  $\mathrm{Ad} \Phi_P$  of  $\mathrm{Ad} P$ . We have

$$L(\gamma, x) = (\mathrm{Ad} \Phi_P)_* : H^1(\Sigma, \mathrm{Ad} P) \rightarrow H^1(\Sigma, \mathrm{Ad} P)$$

if  $\gamma$  is the mapping class of  $\Phi$  and  $x = [P]$ . Introduce the mapping torus

$$(\mathrm{Ad} P)_{\mathrm{Ad} \Phi_P} = (\mathrm{Ad} P \times \mathbb{R}) / ((\mathrm{Ad} \Phi_P)(x), t) \sim (x, t + 1)$$

It is a flat bundle over the mapping torus of  $\Phi$ . It is a well known fact that the Reidemeister torsion of  $(\mathrm{ad} P)_{\mathrm{Ad} \Phi_P}$  satisfies

$$|\tau((\mathrm{ad} P)_{\mathrm{Ad} \Phi_P})| = \frac{1}{|\det(\mathrm{id} - L(\gamma, x))|}$$

We refer the reader to [24] for a proof. The complex numbers  $u(\gamma, x)$  are related to the Chern-Simons invariant of the mapping torus of  $\Phi_P$ . This will be explained in another paper, the difficulty is that the base of this bundle is not a closed manifold, so we must be careful in the definition of the Chern-Simons invariant.

### 3 Hitchin's connection

#### 3.1 Holomorphic differential operators

Let  $M$  be a complex manifold and  $F \rightarrow M$  be a holomorphic line bundle. Consider the algebra of differential operators acting on  $\Gamma(M, F)$ . It is the

direct sum of the subalgebra of holomorphic differential operators and the left-ideal  $\mathcal{I}$  generated by the anti-holomorphic derivations. More explicitly, introduce a local holomorphic trivialization of  $F$  and a system  $(z^i)$  of holomorphic coordinates of  $M$ . Then each differential operator is of the form

$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha \partial_{z^1}^{\alpha(1)} \dots \partial_{z^n}^{\alpha(n)} + \sum_{\alpha, \beta \in \mathbb{Z}^n, \beta \neq 0} a_{\alpha, \beta} \partial_{z^1}^{\alpha(1)} \dots \partial_{z^n}^{\alpha(n)} \partial_{\bar{z}^1}^{\beta(1)} \dots \partial_{\bar{z}^n}^{\beta(n)},$$

where the coefficients  $a_\alpha$  and  $a_{\alpha, \beta}$  are smooth functions. The first summand is a holomorphic differential operator and the second one belongs to the ideal  $\mathcal{I}$ .

We denote by  $D_k^{\text{hol}}(F)$  the bundle whose sections are the holomorphic differential operators of order  $k$  acting on  $\Gamma(M, F)$ .  $D_k^{\text{hol}}(F)$  has a natural holomorphic structure, such that its holomorphic sections are the holomorphic differential operators with holomorphic coefficients. Observe that for any holomorphic differential operator  $P$  and smooth section  $Z$  of  $T^{1,0}M$ , one has

$$D_{\bar{Z}}P = [D_{\bar{Z}}, P] \mod \mathcal{I} \quad (3)$$

where  $D_{\bar{Z}}$  denote the derivative of sections of  $D_k^{\text{hol}}(F)$  (resp.  $F$ ) on the left hand side (resp. right hand side).

### 3.2 Variations of complex structures

Let  $U$  and  $M$  be two manifolds. Consider a smooth family  $(j_u)_{u \in U}$  of complex structures of  $M$ . Denote by  $M_u$  the complex manifold  $(\{u\} \times M, j_u)$ . Let  $E$  be the complex vector bundle over  $U \times M$  with fibers

$$E_{u,x} = T_x^{1,0}M_u.$$

We call  $E$  the relative holomorphic tangent bundle. We shall often consider the decomposition of the tangent space of  $U \times M$  given by

$$T_{u,x}(U \times M) \otimes \mathbb{C} = (T_u U \otimes \mathbb{C}) \oplus E_{u,x} \oplus \bar{E}_{u,x}$$

Let  $X$  be a vector field of  $U$ . Since  $j_u^2 = -\text{id}$ , the derivative of  $(j_u)$  with respect to  $X$  has the form

$$X.j = \mu(X) + \bar{\mu}(X)$$

where  $\mu$  is a section of  $\text{Hom}(\bar{E}, E)$ . Let  $Z$  be a smooth section of  $E$ . Consider  $Z$  and  $X$  as vector fields of  $U \times M$ . Observe that the Lie bracket  $[X, \bar{Z}]$  is tangent to  $E \oplus \bar{E}$ . Furthermore,

$$[X, \bar{Z}] = \frac{i}{2}\mu(X)(\bar{Z}) \mod \bar{E}. \quad (4)$$

which follows easily from the fact that the projection from  $E \oplus \bar{E}$  onto  $\bar{E}$  is  $\frac{1}{2}(\text{id} + ij)$ .

### 3.3 Connection

Consider as previously two manifolds  $U$  and  $M$  with a smooth family  $(j_u)_{u \in U}$  of complex structures. Let  $(F_u \rightarrow M_u)_{u \in U}$  be a smooth family of holomorphic line bundles. Assume that  $M$  is compact and that the dimension of the space  $H^0(M_u, F_u)$  of holomorphic sections does not depend on  $u$ . Then by elliptic regularity, there exists a smooth vector bundle  $\mathcal{H}$  with base  $U$  and fibers  $H^0(M_u, F_u)$ , such that its smooth sections are the smooth families of holomorphic sections (cf [8], chapter 9.2).

Let  $F$  and  $D_2^{\text{hol}}(F)$  be the bundles over  $U \times M$  which restrict over any slice  $M_u$  to  $F_u$  and  $D_2^{\text{hol}}(F_u)$  respectively. If  $s$  is a section of  $F$ , we denote by  $\bar{\partial}s$  the section of  $F \otimes \bar{E}^*$  which restricts over  $M_u$  to  $\bar{\partial}_u s_u$ . So a smooth section of  $\mathcal{H}$  is by definition a smooth section of  $F$  satisfying  $\bar{\partial}s = 0$ . We use the same notation with  $D_2^{\text{hol}}(F)$  and more generally with any family of holomorphic bundles.

Introduce a connection  $\nabla$  on  $F$ , such that its restriction to any  $M_u$  is compatible with the holomorphic structure of  $F_u$ . Introduce a section  $P$  of  $D_2^{\text{hol}}(F) \otimes p^*(T^*U)$  where  $p$  is the projection from  $U \times M$  to  $U$ . We would like to define a connection on  $\mathcal{H} \rightarrow U$  whose covariant derivative in the direction of  $X \in \Gamma(U, TU)$  is given by

$$\nabla_X + P(X). \quad (5)$$

The following lemma provides a sufficient condition for the connection to be well-defined. We denote by  $R^\nabla$  the curvature of  $\nabla$ .

**Proposition 3.1.** *Assume that for any section  $Z$  of  $E$ , we have*

$$[\bar{\partial}P(X)](\bar{Z}) = \frac{i}{2}\nabla_{\mu(X)(\bar{Z})} + R^\nabla(X, \bar{Z})$$

*Then if  $s$  is a section of  $F$  whose restriction to each  $M_u$  is holomorphic, the same holds for  $(\nabla_X + P(X))s$ .*

*Proof.* We show that the assumption implies that for any smooth section  $s$  of  $F$  and any point  $u$  of  $U$

$$([\nabla_{\bar{Z}}, \nabla_X + P(X)]s)_u = Q_u s_u$$

with  $Q_u$  a differential operator of  $F_u$  in the ideal  $\mathcal{I}_u$  generated by the anti-holomorphic derivations. By (4), we have that

$$[\nabla_{\bar{Z}}, \nabla_X] = -\frac{i}{2} \nabla_{\mu(X)(\bar{Z})} - R^\nabla(X, \bar{Z}) \mod \mathcal{I}$$

By (3), we have that

$$[\nabla_{\bar{Z}}, P(X)] = [\bar{\partial}P(X)](\bar{Z}) \mod \mathcal{I}.$$

The conclusion follows.  $\square$

### 3.4 Unicity

Let us discuss the unicity of a connection of the form (5) and satisfying the assumption of proposition 3.1. Assume that  $M$  is connected and that for any  $u$ ,  $M_u$  has no holomorphic vector field. Suppose that  $(\nabla, P)$  satisfies the hypothesis of proposition 3.1 for any vector field  $X$ . Let  $(\nabla', P')$  be another pair satisfying the same assumption. Assume that for any  $u \in U$  and  $X \in T_u U$ ,  $P(X)_u$  and  $P'(X)_u$  are second-order differential operators with the same principal symbol. Then there exists a form  $\alpha \in \Omega^1(U)$  such that

$$\nabla'_X + P'(X) = \nabla_X + P(X) + \alpha(X) \text{id} \quad (6)$$

for any vector field  $X$  of  $U$ .

Indeed,  $\nabla$  and  $\nabla'$  differ by a one-form  $\beta \in \Omega^1(U \times M, \text{End } F)$  which vanishes in the directions tangent to  $\bar{E}$ . Let  $\tilde{\beta}$  be the section of  $\text{End } F \otimes p^*T^*U$  such that  $\beta(X) = \tilde{\beta}(X)$ , for any vector field  $X$  of  $U$ . Then it is easily checked that the pair  $(\nabla' - \beta, P' + \tilde{\beta})$  satisfies the assumption of proposition 3.1. Since

$$\nabla'_X + P'(X) = \nabla'_X - \beta(X) + P'(X) + \tilde{\beta}(X) = \nabla_X + P'(X) + \tilde{\beta}(X)$$

we may assume that  $\nabla = \nabla'$ . Next, the hypothesis of proposition 3.1 implies that

$$\bar{\partial}(P(X) - P'(X)) = 0 \quad (7)$$

Since  $P(X)_u$  and  $P'(X)_u$  have the same principal symbol,  $P(X)_u - P'(X)_u$  is a first order holomorphic differential operator. Since  $H^0(M_u, E_u) = 0$ , equation (7) implies that  $P(X)_u - P'(X)_u$  is a zero-order holomorphic differential operator. In other words,  $P(X)_u - P'(X)_u$  is the multiplication by a function. By equation (7), this function is holomorphic, hence constant because  $M$  is compact and connected. This proves that  $P' = P + \alpha$  with  $\alpha \in \Omega^1(U)$ . The desired equation (6) follows.



### 3.5 A preliminary computation

To apply proposition 3.1, we need to compute the  $\bar{\partial}$  of some second order differential operator. The parameter space doesn't enter in the calculation, so we assume in this subsection that  $U = \{pt\}$ . Suppose that the complex manifold  $M$  has a Kähler metric, and that the holomorphic line bundle  $F$  has a Hermitian metric. Hence  $F$  and  $T^{1,0}M$  have canonical connections compatible with the metric and the holomorphic structure (Chern connection). Let  $G$  be a section of the second symmetric tensor power  $S^2(T^{1,0}M)$ . Define the holomorphic differential operator acting on the sections of  $F$

$$\Delta^G s = \text{Tr}^{\text{End}(T^{1,0}M)}(\nabla^{T^{1,0}M \otimes F}(G \lrcorner \nabla^F s)) \quad (8)$$

More explicitly if  $\partial_1, \dots, \partial_n$  is a local frame of  $T^{1,0}M$  and  $\ell_1, \dots, \ell_n$  is the dual frame,

$$\Delta^G s = \sum_k \ell_k(\nabla_k^{T^{1,0}M \otimes F}(\sum_{i,j} G_{ij} \partial_i \otimes \nabla_j^F s))$$

where  $\nabla_i$  is the covariant derivative with respect to  $\partial_i$  and  $G = \sum G_{ij} \partial_i \otimes \partial_j$ .

**Proposition 3.2.** *Assume that  $G$  is a holomorphic section of  $S^2(T^{1,0}M)$ , then*

$$\bar{\partial} \Delta^G = \sum_{i,j} (2R^F + R^{\det})(\cdot, \partial_i) G_{ij} \nabla_j^F + \theta^F$$

where  $R^F$  is the curvature of  $\nabla^F$ ,  $R^{\det}$  is the curvature of the Chern connection of  $\wedge^n T^{1,0}M$  and  $\theta^F$  is the one-form of  $M$  given by

$$\theta^F(\bar{Z}) = \sum_{k,i,j} \ell_k(\nabla_k^{T^{1,0}M}(R^F(\bar{Z}, \partial_i) G_{ij} \partial_j))$$

for any (local) holomorphic section  $Z$  of  $T^{1,0}M$ .

This is a slight generalization of a computation in [22], page 364. The assumption that the metric is Kähler is used for certain symmetries of the curvature tensor of  $T^{1,0}M$ .

### 3.6 The setting

Assume now that  $M$  is a symplectic manifold and that  $(j_u)_{u \in U}$  is a family of compatible complex structures. So each  $M_u$  is a Kähler manifold.

Let  $L_M \rightarrow M$  be a prequantum bundle, that is a Hermitian line bundle with a connection of curvature  $\frac{1}{i}\omega$ . For any  $u$ , denote by  $L_u \rightarrow M_u$  the

bundle  $L_M$  with the holomorphic structure compatible with the connection and  $j_u$ . Denote by  $L$  the pull-back of  $L_M$  by the projection  $U \times M \rightarrow M$  and endow  $L$  with the pull-back connection.

Consider a pair  $(\delta, \varphi)$  which consists of a line bundle  $\delta$  over  $U \times M$  with an isomorphism  $\varphi$  from  $\delta^2$  to  $\wedge^{\text{top}} E^*$ . Such a pair exists if and only if the second Stiefel-Whitney class of  $M$  vanishes. The restriction  $\delta_u$  of  $\delta$  to  $M_u$  has holomorphic and Hermitian structures determined by the condition that the isomorphism  $\varphi_u : \delta_u^2 \rightarrow \wedge^{\text{top}} E_u^*$  is an isomorphism of Hermitian holomorphic bundles. We call  $(\delta_u, \varphi_u)_{u \in U}$  a family of half-form bundles.

Let us define a connection on  $\delta$ . First consider the connection  $\nabla^{E \oplus \bar{E}}$  on  $E \oplus \bar{E}$  such that its restriction to each slice  $M_u$  is the Levi-Civita connection of the Kähler metric of  $M_u$  and the covariant derivative in a direction tangent to  $U$  is the obvious one. This makes sense because the restriction of  $E \oplus \bar{E}$  to  $U \times \{x\}$  is the trivial bundle with fiber  $T_x M \otimes \mathbb{C}$ . Next we consider the following connection on  $E$

$$\nabla^E = \pi \circ \nabla^{E \oplus \bar{E}},$$

where  $\pi = \frac{1}{2}(\text{id} - ij)$  is the projection of  $E \oplus \bar{E}$  onto  $E$  with kernel  $\bar{E}$ . This defines a connection on the associated bundle  $\wedge^{\text{top}} E^*$  and finally a connection  $\nabla^\delta$  on  $\delta$ .

Our aim is to apply the construction of chapter 3.3 to the bundle  $F = L^k \otimes \delta$ . So consider the vector space

$$\mathcal{H}_{k,u} = H^0(M_u, L_u^k \otimes \delta_u)$$

with the scalar product obtained by integrating the punctual Hermitian product of sections against the Liouville measure of  $M$ . For any compact set  $C$  of  $U$ , if  $k$  is sufficiently large, the dimension of  $\mathcal{H}_{k,u}$  is constant when  $u$  runs over  $C$ . Here we do the global assumption that there exists  $k_0$  such that the dimension of  $\mathcal{H}_{k,u}$  does not depend on  $u \in U$  when  $k \geq k_0$ . For  $k \geq k_0$ , the spaces  $\mathcal{H}_{k,u}$  are the fibers of a smooth vector bundle  $\mathcal{H}_k$  over  $U$ , whose sections are the smooth families of holomorphic sections. In the sequel we always assume that  $k \geq k_0$ .

### 3.7 Existence of the connection

Consider the same data as in section 3.6. To define a connection on the bundle  $\mathcal{H}_k \rightarrow U$ , we assume first that  $M$  is simply connected. Then for any tangent vector  $X$  of  $U$  at  $u$ , introduce the section  $G(X)$  of  $S^2(T^{1,0} M_u)$  such

that

$$\sum_{i,j} G_{ij} \omega(\partial_i, \cdot) \partial_j = \mu(X)_u, \quad G(X) = G_{ij} \partial_i \otimes \partial_j \quad (9)$$

Our second assumption is that these sections  $G(X)$  are holomorphic.

To apply proposition 3.1 to the bundle  $F = L^k \otimes \delta$ , we need to compute the curvature  $R^\delta$  of  $\nabla^\delta$ . The following result is proved in [4].

**Proposition 3.3.** *For any tangent vector  $X \in T_u U$  and section  $Z$  of  $E$ , we have*

$$4R^\delta(X, \bar{Z}) = \theta^{L_u}(\bar{Z})$$

where  $\theta^{L_u}$  is defined as in proposition 3.2.

As a consequence of proposition 3.2, we have for any tangent vector  $X \in T_u U$  and section  $Z$  of  $E_u$ ,

$$\begin{aligned} [\bar{\partial} \Delta^{G(X)}](\bar{Z}) &= 2ik \nabla_{\mu(X)(\bar{Z})}^{L^k \otimes \delta} + k \theta^{L_u}(\bar{Z}) + \theta^{\delta_u}(\bar{Z}) \\ &= 4k \left[ \frac{i}{2} \nabla_{\mu(X)(\bar{Z})}^{L^k \otimes \delta} + R^\delta(X, \bar{Z}) \right] + \theta^{\delta_u}(\bar{Z}) \end{aligned}$$

by proposition 3.3. For  $k = 0$ , this implies that  $\bar{\partial} \theta^{\delta_u} = 0$ . Since  $M$  is simply connected, by Hodge decomposition, the Dolbeault cohomology group  $H^{1,0}(M_u)$  vanishes for any  $u$ . So there exists a function  $H(X)$  such that

$$\bar{\partial} H(X) = \theta^{\delta_u}, \quad \int_M H(X) \omega^n = 0 \quad (10)$$

This defines a section  $H \in \Gamma(U \times M, p^*(T^*U))$ . Then

$$\nabla_X^{L^k \otimes \delta} + \frac{1}{4k} (\Delta^{G(X)} - H(X))$$

satisfies the assumption of proposition 3.1. To summarize, we have prove the following theorem. This was the main theorem of [4].

**Theorem 3.4.** *Let  $M$  be a simply-connected compact symplectic manifold endowed with a prequantum bundle  $L_M$  and a family  $(j_u, \delta_u, \varphi_u)_{u \in U}$  of compatible complex structures with half-form bundles. Assume that the sections  $G(X)$ ,  $X \in TU$ , defined in (9) are holomorphic. Then the bundle  $\mathcal{H}_k \rightarrow U$  with fibers  $\mathcal{H}_{k,u} = H^0(M_u, L_u^k \otimes \delta_u)$  admits a connection  $\nabla^{\mathcal{H}_k}$  defined by*

$$\nabla_X^{\mathcal{H}_k} := \nabla_X^{L^k \otimes \delta} + \frac{1}{4k} (\Delta^{G(X)} - H(X)), \quad X \in \Gamma(U, TU) \quad (11)$$

where  $\Delta^{G(X)}$  and  $H(X)$  are given by the equations (8) and (10)

### 3.8 Application to the moduli space

We can apply the previous constructions to the moduli space  $\mathcal{M}$  and the Chern-Simons bundle, cf. chapter 1.1, with the Teichmüller space as a parameter space. All the assumptions we made are satisfied. We obtain a connection explicitly given by (11). We assert that this connection is projectively flat. This can be deduced from the results of [22] and our discussion about the unicity.

Indeed, in [22], Hitchin considered the quantum space  $\mathcal{H}'_k$  over  $\mathcal{T}$  whose fiber at  $\sigma$  is  $H^0_\sigma(\mathcal{M}_\sigma, L^k_\sigma)$ . He proved the existence of a projectively flat connection of the form

$$\nabla_X^{L^k} + P'_k(X), \quad X \in \Gamma(\mathcal{T}, T\mathcal{T})$$

where for any  $\sigma$ ,  $P'_k(X)_\sigma$  is a second order differential operator. Furthermore the symbol of  $P'_k(X)_\sigma$ , viewed as a section of  $S^2(E_\sigma)$ , is  $G/(4(k+1))$ . Because of the isomorphism  $\delta \simeq L^{-1}$ , this defines a connection on  $\mathcal{H}_{k+1}$ . It follows from the discussion of chapter 3.4 that this connection is the same as (11) up to a term of the form  $\alpha \text{id}$  with  $\alpha \in \Omega^1(\mathcal{T})$ . So the curvatures of the two connections differ by  $d\alpha \text{id}$ .

## 4 An algebra of Toeplitz operators

Consider a complex compact manifold  $M$  and a family of Hermitian holomorphic line bundles  $F = (F_j \rightarrow M)_{j \in J}$ . For any  $j \in J$ , let  $\text{Op}(F_j)$  be the algebra of holomorphic differential operators acting on the sections of  $F_j$ . Consider the subalgebra  $\text{Op}_{sc}(F)$  of  $\prod_{j \in J} \text{Op}(F_j)$  consisting of the family  $(P_j)$  satisfying the following condition: there exists  $\ell$  such that for any complex coordinate system  $(U, z^1, \dots, z^n)$ , there exists a family  $(a_\alpha)_{|\alpha| \leq \ell}$  of  $\mathcal{C}^\infty(U)$  such that we have over  $U$

$$P_j = \sum_{\alpha} a_\alpha (\nabla_1^{F_j})^{\alpha(1)} \dots (\nabla_n^{F_j})^{\alpha(n)}, \quad \forall j \in J.$$

Here  $\nabla^{F_j}$  is the Chern connection of  $F_j$ , and  $\nabla_\ell^{F_j}$  is the covariant derivative with respect to  $\partial_{z^\ell}$ . To check that  $\text{Op}_{sc}(F)$  is a subalgebra, it suffices to use that  $[\nabla_k^{F_j}, \nabla_\ell^{F_j}] = 0$ .

Let  $\mu$  be a measure of  $M$ . We define a scalar product on  $\Gamma(M, F_j)$  by integrating the pointwise scalar product of sections against  $\mu$ . Denote by  $\Pi_j$  the orthogonal projector of  $\Gamma(M, F_j)$  onto its subspace of holomorphic sections  $H^0(M, F_j)$ .

**Proposition 4.1.** *For any  $(P_j) \in \text{Op}_{sc}(F)$ , there exists a function  $f \in \mathcal{C}^\infty(M)$  such that for any  $j \in J$*

$$\Pi_j P_j \Pi_j = \Pi_j M_f \Pi_j$$

where  $M_f$  is multiplication operator of  $\Gamma(M, F_j)$  with multiplier  $f$ .

The proof is based on a trick due to Tuynman [31], a similar result was also used in [2].

*Proof.* Let  $(P_j) \in \text{Op}(F)$ . Observe that all the  $P_j$  have the same order  $\ell$  and the same symbol  $\sigma$ . This symbol is a section of  $S^\ell(T^{1,0}M)$ . Using a partition of unity, we can write  $\sigma$  under the form

$$\sigma = \sum_{i=1, \dots, r} X_1^i \otimes \dots \otimes X_\ell^i$$

where the  $X_k^i$  are smooth sections of  $T^{1,0}M$ . Since

$$P_j - \sum_{i=1, \dots, r} \nabla_{X_1^i}^{F_j} \dots \nabla_{X_\ell^i}^{F_j}$$

has order  $\ell - 1$ , it suffices to prove the proposition for each operator

$$\nabla_{X_1^i}^{F_j} \dots \nabla_{X_\ell^i}^{F_j}.$$

Let  $s_1, s_2$  be smooth sections of  $F_j$ . Assume that  $s_2$  is holomorphic. Then for any smooth section  $X$  of  $T^{1,0}M$ , we have

$$X \cdot [(s_1, s_2)\mu] = (\nabla_X^{F_j} s_1, s_2)\mu + (s_1, s_2)(\text{div } X)\mu$$

So if we denote by  $(\cdot, \cdot)_{F_j}$  the scalar product of sections we obtain

$$(\nabla_X^{F_j} s_1, s_2)_{F_j} = (f s_1, s_2)_{F_j}$$

with  $f = -\text{div } X$ . So if  $X_1, \dots, X_\ell$  are smooth sections of  $T^{1,0}M$ , then

$$\begin{aligned} (\nabla_{X_1}^{F_j} \dots \nabla_{X_\ell}^{F_j} s_1, s_2)_{F_j} &= (f_1 \nabla_{X_2}^{F_j} \dots \nabla_{X_\ell}^{F_j} s_1, s_2)_{F_j} \\ &= (\nabla_{f_1 X_2}^{F_j} \dots \nabla_{X_\ell}^{F_j} s_1, s_2)_{F_j} \\ &\dots \\ &= (f s_1, s_2)_{F_j} \end{aligned}$$

where  $f$  is a function which depends only on  $\mu$  and the vector fields  $X_i$ . This proves the result.  $\square$

## 5 Asymptotic flatness

In this part we consider the same data as in section 3.6. We will prove that the curvature of the connection  $\nabla^{\mathcal{H}_k}$  defined in theorem 3.4 vanishes in the semi-classical limit  $k \rightarrow \infty$ . For any vector field  $X$  of  $U$ , denote by  $P_k(X)$  the operator

$$P_k(X) = \frac{1}{4}(\Delta^{G(X)} - H(X))$$

The curvature of (11) in the directions  $X, Y \in \Gamma(U, TU)$  is

$$\begin{aligned} R_k(X, Y) = & [\nabla_X^{L^k \otimes \delta} + k^{-1}P_k(X), \nabla_Y^{L^k \otimes \delta} + k^{-1}P_k(Y)] \\ & - \nabla_{[X, Y]}^{L^k \otimes \delta} - k^{-1}P_k([X, Y]). \end{aligned}$$

Recall that the algebra of differential operators acting on the sections of a holomorphic fiber bundle is the direct sum of the algebra of holomorphic differential operators and the left ideal generated by the anti-holomorphic vector fields. If  $Q_u$  is a differential operator acting on  $\Gamma(M_u, L_u^k \otimes \delta_u)$ , we denote by  $Q_u^{\text{Hol}}$  its holomorphic part. We use the same notations for families  $(Q_u)_{u \in U}$ . We denote by  $\text{Op}_{sc}$  the space of families

$$(Q_{u,k} : \Gamma(M_u, L_u^k \otimes \delta_u) \rightarrow \Gamma(M_u, L_u^k \otimes \delta_u))_{u \in U, k \geq k_0}$$

consisting of differential operators such that for any  $k$ ,  $Q_{u,k}$  depends smoothly on  $u$  and for any  $u$ ,  $(Q_{u,k})_k$  belongs to the algebra  $\text{Op}_{sc}(L_u^k \otimes \delta_u, k \geq k_0)$  introduced in the previous section.

**Theorem 5.1.** *For any vector fields  $X, Y$  of  $U$ , one has*

$$R_k(X, Y)^{\text{Hol}} = k^{-1}P_{1,k}(X, Y) + k^{-2}P_{2,k}(X, Y), \quad \forall k \geq k_0$$

where the families  $(P_{1,k}(X, Y))_k$  and  $(P_{2,k}(X, Y))_k$  belong to  $\text{Op}_{sc}$ .

Theorem 2.1 about the asymptotic flatness follows. Indeed by proposition 4.1, for any family  $(Q_{u,k})$  of  $\text{Op}_{sc}$ , there exists a continuous function  $C : U \rightarrow \mathbb{R}$  such that for any  $k$  and  $u$ , the uniform norm of

$$\Pi_{u,k} Q_{u,k} : H^0(M_u, L_u^k \otimes \delta_u) \rightarrow H^0(M_u, L_u^k \otimes \delta_u)$$

is bounded by  $C(u)$ .

The remainder of this section is devoted to the proof of theorem 5.1. Since  $L$  is the pull-back of a bundle over  $M$ , its curvature in the directions tangent to  $U$  vanishes. For the half-form bundle, the curvature  $R^\delta$  depends on the derivative of the complex structure. Recall that we denote by  $\mu(X)$  the variation of the complex structure, cf. section 3.2.

**Proposition 5.2.** *For any vector field  $X, Y$  of  $U$ ,*

$$R^\delta(X, Y) = \frac{1}{8} \operatorname{tr}(\mu(X)\overline{\mu}(Y) - \mu(Y)\overline{\mu}(X)).$$

The proof is easy, cf. as instance the proof of theorem 7.2 in [15].

**Proposition 5.3.** *For any vector fields  $X, Y$  of  $U$  we have*

$$[\nabla_X^{L^k \otimes \delta}, P_k(Y)]^{\operatorname{Hol}} = \frac{k}{8} \operatorname{tr}(\mu(Y)\overline{\mu}(X)) + P_k(X, Y)$$

where the family  $(P_k(X, Y))$  belongs to  $\operatorname{Op}_{sc}$ .

*Proof.* Introduce a local frame  $\partial_1, \dots, \partial_n$  of the relative holomorphic tangent bundle of  $M \times U$ . Denote by  $\nabla^k$  the covariant derivative of  $L^k \otimes \delta$  and by  $\nabla_i^k$  the covariant derivative in the direction of  $\partial_i$ . In the sequel, repeated indices  $i$  and  $j$  are summed over. We have

$$\Delta^{G(Y)} = f_j \nabla_j^k + G_{i,j} \nabla_i^k \nabla_j^k. \quad (12)$$

where the coefficients  $f_i$  and  $G_{ij}$  do not depend on  $k$ . Then

$$[\nabla_X^k, f_j \nabla_j^k] = (X.f_j) \nabla_j^k + f_j \nabla_{[X, \partial_j]}^k + f_j R^k(X, \partial_j) \quad (13)$$

where  $R^k$  is the curvature of  $\nabla^k$ . The first term of the right hand side clearly belongs to  $\operatorname{Op}_{sc}$ , the third term also because

$$R^k(X, \partial_j) = R^\delta(X, \partial_j)$$

is independent of  $k$ . For the second term, observe that the holomorphic part of  $\nabla_{[X, \partial_j]}^k$  is a linear combination of the  $\nabla_j^k$  with smooth coefficients which do not depend on  $k$ . So the holomorphic part of (13) belongs to  $\operatorname{Op}_{sc}$ . Let us compute the bracket of  $\nabla_X^k$  with the second term of (12).

$$[\nabla_X^k, G_{ij} \nabla_i^k \nabla_j^k] = (X.G_{ij}) \nabla_i^k \nabla_j^k + G_{ij} [\nabla_X^k, \nabla_i^k] \nabla_j^k + G_{ij} \nabla_i^k [\nabla_X^k, \nabla_j^k]$$

The first term of the right hand side belongs to  $\operatorname{Op}_{sc}$ . The same holds for the holomorphic part of the third term because

$$G_{ij} \nabla_i^k [\nabla_X^k, \nabla_j^k] = G_{ij} \nabla_i^k \nabla_{[X, \partial_j]}^k + G_{ij} \nabla_i^k R^k(X, \partial_j).$$

and we can argue as we did for (13). The second term is equal to

$$\begin{aligned} G_{ij} [\nabla_X^k, \nabla_i^k] \nabla_j^k &= G_{ij} \nabla_{[X, \partial_i]}^k \nabla_j^k + G_{ij} R^k(X, \partial_i) \nabla_j^k \\ &= G_{ij} \nabla_j^k \nabla_{[X, \partial_i]}^k + G_{ij} \nabla_{[[X, \partial_i], \partial_j]}^k + G_{ij} R^k([X, \partial_i], \partial_j) \\ &\quad + G_{ij} R^k(X, \partial_i) \nabla_j^k \end{aligned}$$

All the terms of this last sum have a holomorphic part in  $\text{Op}_{sc}$  except the third one which is equal to

$$G_{ij}R^k([X, \partial_i], \partial_j) = G_{ij}R^\delta([X, \partial_i], \partial_j) + \frac{k}{i}G_{ij}\omega([X, \partial_i], \partial_j)$$

Since  $\mu(Y) = G_{ij}\omega(\partial_i, \cdot)\partial_j$ , we have

$$\text{tr}(\mu(Y)\overline{\mu}(X)) = G_{ij}\omega(\partial_i, \overline{\mu}(X)(\partial_j))$$

Using that  $[X, \partial_i] = -\frac{i}{2}\overline{\mu}(X)(\partial_i)$  modulo  $E$ , it follows that

$$\frac{k}{i}G_{ij}\omega([X, \partial_i], \partial_j) = \frac{k}{2}\text{tr}(\mu(Y)\overline{\mu}(X))$$

Collecting the various terms, we obtain the result.  $\square$

Let us conclude the proof of theorem 5.1. We have

$$\begin{aligned} R_k(X, Y) = & R^\delta(X, Y) + \frac{1}{k}[\nabla_X^{L^k \otimes \delta}, P_k(Y)] - \frac{1}{k}[\nabla_Y^{L^k \otimes \delta}, P_k(X)] \\ & + \frac{1}{k^2}[P_k(X), P_k(Y)] - \frac{1}{k}P_k([X, Y]) \end{aligned}$$

By propositions 5.2 and 5.3, the holomorphic part of the sum of the first three terms is in  $k^{-1}\text{Op}_{sc}$ . The last two terms belong respectively to  $k^{-2}\text{Op}_{sc}$  and  $k^{-1}\text{Op}_{sc}$ .

## 6 Semi-classical connection

Let  $(M, \omega)$  be a compact symplectic manifold with a prequantum bundle  $L_M \rightarrow M$ . Consider a manifold  $U$  and a smooth family  $(j_u, \delta_u, \varphi_u)_{u \in U}$  consisting of positive complex structures with half-form bundles.

We adopt the same notations and conventions as in sections 3.2 and 3.6. Namely,  $M_u = \{u\} \times M$  is endowed with the complex structure  $j_u$ ,  $L_u \rightarrow M_u$  is the prequantum bundle with the holomorphic structure induced by  $j_u$ . We denote by  $L$ ,  $\delta$  and  $E$  the bundles over  $U \times M$  whose restrictions to each  $M_u$  are  $L_u$ ,  $\delta_u$  and  $T^{1,0}M_u$ . Let  $\mathcal{H}_k$  be the vector bundle over  $U$  whose fibers are the Hilbert spaces

$$\mathcal{H}_{k,u} = H^0(M_u, L_u^k \otimes \delta_u)$$

and denote by  $\Pi_{k,u}$  the orthogonal projector from  $\Gamma(M_u, L_u^k \otimes \delta_u)$  onto  $\mathcal{H}_{k,u}$ .



We now define a connection of the bundle  $\mathcal{H}_k$ . Consider the same connections on  $\delta$  and  $L$  as in section 3.6. We set for any vector field  $X$  of  $U$  and section  $s$  of  $\mathcal{H}_k$

$$(\nabla_X^{\text{Toep},k} s)(u) := \Pi_{k,u}((\nabla_X^{L^k \otimes \delta} s)(u))$$

It is easily proved that this is indeed a connection. More generally we shall consider the connections

$$\nabla_X^{\text{Toep},k} + A_k, \quad A_k \in \Omega^1(U, \text{End}(\mathcal{H}_k))$$

where the family  $(A_k, k = 1, 2, \dots)$  is a Toeplitz operator. This has the following meaning. Let  $p$  be the projection from  $U \times M$  onto  $U$ . Then there exists a sequence  $f(\cdot, k)$  of  $\Gamma(U \times M, p^*(T^*U \otimes \mathbb{C}))$  admitting an asymptotic expansion of the form  $f_0 + k^{-1}f_1 + \dots$  for the  $\mathcal{C}^\infty$ -topology on the compact subsets, such that

$$A_k(X)(u) = \Pi_{k,u} M_{f(X)(\cdot, u, k)} : \mathcal{H}_{k,u} \rightarrow \mathcal{H}_{k,u}$$

for any vector field  $X$  of  $M$ . Here  $M_g$  denote the multiplication operator by  $g$ . We call  $f_0$  the principal symbol of  $(A_k)$ . This includes the connection defined in section 3.7. Indeed proposition 4.1 implies the

**Proposition 6.1.** *There exists  $f_1 \in \Gamma(U \times M, \pi^*(T^*U) \otimes \mathbb{C})$  such that for any  $k \geq k_0$ , we have*

$$\nabla_X^{\mathcal{H}_k} = \nabla_X^{\text{Toep},k} + k^{-1} \Pi_k M_{f_1(X)}$$

for any vector field  $X$  of  $U$ .

The results in [2] also rely on the comparison between  $\nabla_X^{\mathcal{H}_k}$  and  $\nabla_X^{\text{Toep},k}$  (without the metaplectic correction).

The following theorem says that the curvature of these connections is a Toeplitz operator in a semi-classical sense.

**Theorem 6.2.** *There exists a sequence  $g(\cdot, k) \in \Gamma(U \times M, p^*(\wedge^2 T^*U \otimes \mathbb{C}))$  admitting an asymptotic expansion of the form  $g_0 + k^{-1}g_1 + \dots$  for the  $\mathcal{C}^\infty$ -topology on compact subsets, such that the curvature of  $\nabla_X^{\text{Toep},k} + A_k$  satisfies*

$$R^{A_k}(X, Y)_u = \Pi_{k,u} M_{g(X, Y)(\cdot, u, k)} + O(k^{-\infty})$$

where the  $O(k^{-\infty})$  is uniform on compact set of  $U$ . Furthermore,  $g_0$  is given by

$$g_0(X, Y) = X.f_0(Y) - Y.f_0(X) - f_0([X, Y])$$

with  $f_0$  the principal symbol of  $(A_k)$ .

This theorem is a generalization of theorem 7.1 of [15], where we did not consider the terms  $(A_k)$ . The proof is an immediate generalization of the one in [15]. This provides another proof of theorem 2.1.

To describe the parallel transport along a path  $\gamma$  in the bundle  $\mathcal{H}_k$  we introduce the notion of half-form isomorphism. For any  $u, u' \in U$  and  $x \in M$  denote by  $\pi_{u',u,x}$  the projection from  $E_{u',x}$  to  $E_{u,x}$  with kernel  $\bar{E}_{u',x}$ . We say that a linear isomorphism  $\Psi$  of  $\text{Hom}(\delta_{u,x}, \delta_{u',x})$  is a *half-form isomorphism* if its square is the pull-back by  $\pi_{u',u,x}$ , more precisely

$$\varphi_{u',x} \circ \Psi^2 = \pi_{u',u,x}^* \circ \varphi_{u,x}.$$

Such an isomorphism is unique up to a plus or minus sign. If  $\gamma$  is a path of  $U$ , then for any  $x$ , there exists a unique continuous path of half-form morphism  $\delta_{\gamma(0),x} \rightarrow \delta_{\gamma(t),x}$  starting from the identity. We denote by  $\Psi(\gamma)$  the morphism  $\delta_{\gamma(0)} \rightarrow \delta_{\gamma(1)}$  obtained at  $t = 1$ .

**Theorem 6.3.** *Let  $u, u' \in U$  and  $\gamma$  be a path from  $u$  to  $u'$ . For any  $k \geq k_0$ , let  $T_k : \mathcal{H}_{k,u} \rightarrow \mathcal{H}_{k,u'}$  be the parallel transport along  $\gamma$  in the bundle  $\mathcal{H}_k$  for the connection  $\nabla^{\text{Toep},k} + A_k$ . Then the Schwartz kernels of the operators  $T_k$  have the following form*

$$T_k(x, y) = \left( \frac{k}{2\pi} \right)^n F^k(x, y) \otimes f(x, y, k) + O(k^{-\infty})$$

where  $n$  is half the dimension of  $M$  and

- $F$  is a section of  $L \boxtimes \bar{L}$  such that  $|F(x, y)| < 1$  if  $x \neq y$ ,  $F(x, x) = v \otimes \bar{v}$  for all  $x$  and  $v \in L_x$  of norm 1, and  $\bar{\partial}_{j_{u'} \times -j_u} F \equiv 0$  modulo a section vanishing to any order along the diagonal.
- $f(., k)$  is a sequence of sections of  $\delta_{u'} \boxtimes \bar{\delta}_u \rightarrow M^2$  which admits an asymptotic expansion in the  $C^\infty$  topology of the form

$$h(., k) = h_0 + k^{-1}h_1 + k^{-2}h_2 + \dots$$

whose coefficients satisfy  $\bar{\partial}_{j_{u'} \times -j_u} f_i \equiv 0$  modulo a section vanishing to any order along the diagonal.

- If the principal symbol of  $(A_k)$  vanishes, then  $h_0(x, x) = \Psi(\gamma).v \otimes \bar{v}$  for any  $x \in M$  and  $v \in \delta_{u,x}$  with norm 1.

For  $A_k = 0$ , this is theorem 7.1 of [15]. In the next section we provide a proof of theorem 6.3 more direct than the one in [15] and which shows clearly the role of the half-form bundle.

## 7 Parallel transport

This section is devoted to the proof of theorem 6.3. As previously, assume that  $(M, \omega)$  is a symplectic manifold with a prequantum bundle  $L_M$ . Choose an open interval  $I \subset \mathbb{R}$  for the parameter space. Introduce a smooth family  $(j_t)_{t \in I}$  of positive complex structures of  $(M, \omega)$ . We use the same notation as before. For instance,  $M_t$  is the Kähler manifold  $(M, \omega, j_t)$  and  $L_t$  is the prequantum bundle with its holomorphic structure induced by  $j_t$ . Instead of half-form bundles we begin with a smooth family  $(H_t \rightarrow M_t)_{t \in I}$  of holomorphic Hermitian line bundles. Denote by  $\mathcal{H}_{k,t}$  the space  $H^0(M_t, L_t^k \otimes H_t)$  and by  $\Pi_{k,t}$  the orthogonal projector from  $\Gamma(M_t, L_t^k \otimes H_t)$  onto  $\mathcal{H}_{k,t}$ .

Let  $\Lambda$  be a closed Lagrangian submanifold of  $(M, \omega)$  and  $s$  be a flat unitary section of the restriction of the prequantum bundle to  $\Lambda$ . Introduce a section  $F$  of  $L \rightarrow I \times M$  such that

$$F(t, x) = s(x), \quad \forall x \in \Lambda$$

the pointwise norm of  $F$  is  $< 1$  outside  $I \times \Lambda$  and  $\bar{\partial}F$  vanishes to any order along  $I \times \Lambda$ . It is not obvious but nevertheless true that such a section exists (proposition 2.1 of [13]). It is unique up to a section vanishing to any order along  $\Lambda$ .

We say that a sequence  $(f(\cdot, k))_k$  of  $\Gamma(I \times M, H)$  is a symbol if it admits an asymptotic expansion for the  $\mathcal{C}^\infty$  topology on compact subsets of the form  $f_0 + k^{-1}f_1 + \dots$  with coefficients in  $\Gamma(I \times M, H)$ . We call  $f_0$  the leading coefficient of  $f(\cdot, k)$  even if it vanishes. Recall the following basic result (lemma 2.5 of [13]).

**Theorem 7.1.** *For any symbol  $f(\cdot, k)$  of  $\Gamma(I \times M, H)$  we have*

$$\Pi_k(F^k f(\cdot, k)) = F^k g(\cdot, k) + O(k^{-\infty})$$

where  $g(\cdot, k)$  is a symbol of  $\Gamma(I \times M, H)$ . Furthermore the restrictions to  $\Lambda$  of the leading coefficients of  $f(\cdot, k)$  and  $g(\cdot, k)$  are equal.

Another important fact is that a family  $(\Psi_k \in \Gamma(U, \mathcal{H}_k))$  of the form

$$\Psi_k = F^k f(\cdot, k) + O(k^{-\infty})$$

is determined by the restriction of the symbol  $f(\cdot, k)$  to  $I \times \Lambda$ . Indeed one proves that  $\Psi_k$  is  $O(k^{-\infty})$  if and only if the restriction of  $f(\cdot, k)$  to  $I \times \Lambda$  is  $O(k^{-\infty})$  (lemma 1 of [12])

The next theorem involves the function  $c$  of  $I \times \Lambda$  defined by

$$c(t, x) = \frac{1}{4} \sum_{i=1}^n \omega(\mu_{t,x}(\bar{\partial}_i), \bar{\partial}_i), \quad (t, x) \in I \times \Lambda \quad (14)$$

Here  $(\partial_1, \dots, \partial_n)$  is a basis of  $T_x^{1,0} M_t$  such that  $\partial_i + \bar{\partial}_i$  is tangent to  $\Lambda$  and  $\frac{1}{i} \omega(\partial_i, \bar{\partial}_j) = \delta_{ij}$  for any indices  $i$  and  $j$ . As in section 3.2,  $\mu$  is the map such that

$$\frac{d}{dt} j_t = \mu_{t,x} + \bar{\mu}_{t,x}, \quad \mu_{t,x} : T_x^{0,1} M_t \rightarrow T_x^{1,0} M_t$$

Introduce a connection on the bundle  $H \rightarrow I \times M$ .

**Theorem 7.2.** *For any symbol  $f(\cdot, k)$  of  $\Gamma(I \times M, H)$ , we have that*

$$\Pi_k(\nabla_{\partial_t}^{L^k \otimes H}(F^k f(\cdot, k))) = F^k g(\cdot, k) + O(k^{-\infty})$$

where  $g(\cdot, k)$  is a symbol of  $\Gamma(I \times M, H)$ . Furthermore, the leading coefficients  $f_0$  and  $g_0$  satisfy

$$g_0(t, x) = (\nabla_{\partial_t}^H f_0)(t, x) - c(t, x) f_0(t, x), \quad \forall (t, x) \in I \times \Lambda.$$

*Proof.* By proposition 9.4 in [15], we have  $\nabla_{\partial_t}^L F = hF$  where  $h$  is a function on  $I \times M$  that vanishes along  $I \times \Lambda$ . The derivatives of  $h$  also vanish along  $I \times \Lambda$  and the second derivatives satisfy

$$\bar{Z}_1 \cdot \bar{Z}_2 \cdot h(t, x) = -\frac{1}{2} \omega(\bar{Z}_1, \mu(\bar{Z}_2)), \quad (t, x) \in I \times \Lambda$$

for any vectors  $Z_1, Z_2 \in T_x^{1,0} M_t$ . Then we have

$$\nabla_{\partial_t}^{L^k \otimes H}(F^k f(\cdot, k)) = F^k(kh f(\cdot, k) + \nabla_{\partial_t}^H f(\cdot, k))$$

Next by theorem 4.1 of [14] and theorem 7.1, we have

$$\Pi_k \nabla_{\partial_t}^{L^k \otimes H}(F^k f(\cdot, k)) = F^k g(\cdot, k) + O(k^{-\infty})$$

for a symbol  $g(\cdot, k) = g_0 + O(k^{-1})$  with

$$g_0 = -\frac{1}{2} f_0 \sum_{i=1}^n \bar{\partial}_i \bar{\partial}_i h + \nabla_{\partial_t}^H f_0$$

at any  $(t, x) \in I \times \Lambda$ . Here the basis  $(\partial_i)_i$  of  $T_x^{1,0} M_t$  satisfies the same conditions as the one in the definition of  $c(t, x)$ .  $\square$

Let  $\pi = \frac{1}{2}(\text{id} - ij)$  be the projection of  $E \oplus \bar{E}$  onto  $E$  with kernel  $\bar{E}$ . For any  $(t, x) \in I \times \Lambda$ ,  $\pi_{t,x}$  restricts to an isomorphism from  $T_x \Lambda \otimes \mathbb{C}$  onto  $T_x^{1,0} M_t$ . So the bundles  $p^*(T\Lambda \otimes \mathbb{C})$  and  $\iota^* E$  are naturally isomorphic, where  $p$  and  $\iota$  are respectively the projection  $I \times \Lambda \rightarrow \Lambda$  and the injection  $I \times \Lambda \rightarrow I \times M$ . Consequently

$$p^* \wedge^{\text{top}} (T^* \Lambda \otimes \mathbb{C}) \simeq \iota^* \wedge^{\text{top}} E^* \quad (15)$$

Denote by  $D_t$  the obvious derivation of the sections of  $p^* \wedge^{\text{top}} (T^* \Lambda \otimes \mathbb{C})$  with respect to  $\partial_t$ . Recall that  $\nabla_{\partial_t}^E$  is the derivative of  $\Gamma(I \times M, E)$  obtained by composing the obvious derivation in  $E \oplus \bar{E}$  with respect to  $\partial_t$  with the projection  $\pi$ .

**Proposition 7.3.** *Identifying the sections of  $\iota^* \wedge^{\text{top}} E^*$  with the sections of  $p^* \wedge^{\text{top}} (T^* \Lambda \otimes \mathbb{C})$  through the isomorphism (15) induced by  $\pi$ , we have*

$$D_t = \nabla_{\partial_t}^{\iota^* (\wedge^{\text{top}} E^*)} - 2c$$

with  $c$  the function defined in (14).

*Proof.* Let  $x$  be a point of  $\Lambda$ . Consider a smooth curve  $t \rightarrow U(t)$  of  $T_x \Lambda$ . We have

$$\nabla_{\partial_t}^E (\pi_{t,x} U(t)) = \pi_{t,x} (\dot{\pi}_{t,x} U(t) + \pi_{t,x} \dot{U}(t)).$$

Using that  $\pi = \frac{1}{2}(\text{id} - ij)$ , we prove that

$$\dot{\pi}_{t,x} = -\frac{i}{2}(\mu_{t,x} \circ \bar{\pi}_{t,x} + \bar{\mu}_{t,x} \circ \pi_{t,x})$$

Consequently

$$\nabla_{\partial_t}^E (\pi_{t,x} U(t)) = -\frac{i}{2} \mu_{t,x} \circ \bar{\pi}_{t,x} (U(t)) + \pi_{t,x} (\dot{U}(t))$$

To end the proof we have to show that

$$\frac{i}{2} \text{tr}(\mu_{t,x} \circ \bar{\pi}_{t,x} \circ q_{t,x} : T_x^{1,0} M_t \rightarrow T_x^{1,0} M_t) = 2c(t, x)$$

where  $q_{t,x} : T_x^{1,0} M \rightarrow T_x \Lambda \otimes \mathbb{C}$  is the inverse of the restriction of  $\pi_{t,x}$  to  $T_x \Lambda \otimes \mathbb{C}$ . Introduce a basis  $\partial_1, \dots, \partial_n$  of  $T_x^{1,0} M \otimes \mathbb{C}$  as in the definition (14) of  $c$ . Then  $q_{t,x}(\partial_i) = \partial_i + \bar{\partial}_i$ , so that  $\bar{\pi}_{t,x}(q_{t,x}(\partial_i)) = \bar{\partial}_i$  and then

$$\begin{aligned} \text{tr}(\mu_{t,x} \circ \bar{\pi}_{t,x} \circ q_{t,x} : T_x^{1,0} M_t \rightarrow T_x^{1,0} M_t) &= \frac{1}{i} \sum_j \omega(\mu_{t,x}(\bar{\partial}_j), \bar{\partial}_j) \\ &= \frac{4}{i} c(t, x) \end{aligned}$$

which concludes the proof.  $\square$

Let us consider a family of half-form bundles  $(\delta_t, \varphi_t)$ . For any  $(t, x) \in I \times \Lambda$ , one has an isomorphism

$$\tilde{\varphi}_{t,x} = \pi_{t,x}^* \varphi_{t,x} : \delta_{t,x}^{\otimes 2} \rightarrow \wedge^{\text{top}}(T\Lambda \otimes \mathbb{C})^*$$

So we have a natural derivation  $D_t$  of the sections of  $\iota^* \delta$  satisfying

$$D_t \tilde{\varphi}(s^{\otimes 2}) = 2\tilde{\varphi}(s \otimes D_t s).$$

Let us apply theorem 7.2 with the family  $(H_t) = (\delta_t)$  and the connection  $\nabla^\delta$  induced by  $\nabla^E$ . Then by the last proposition, the equation determining the leading coefficient is

$$\iota^* g_0 = D_t(\iota^* f_0).$$

**Theorem 7.4.** *Let  $f(\cdot, k)$  be a symbol of  $\mathcal{C}^\infty(I \times M)$ . Let  $(\Psi_k \in \Gamma(I, \mathcal{H}_k))_k$  be a family satisfying*

$$\Pi_k(\nabla_{\partial_t}^{L^k \otimes \delta} \Psi_k + M_{f(\cdot, k)} \Psi_k) = 0$$

*Assume that there exists  $t_0 \in I$  and a symbol  $g(\cdot, k)$  of  $\Gamma(M, \delta_{t_0})$  such that  $\Psi_k(t_0) = F^k(t_0, \cdot)g(\cdot, k) + O(k^{-\infty})$ . Then there exists a symbol  $h(\cdot, k)$  of  $\Gamma(I \times M, \delta)$  such that*

$$\Psi_k = F^k h(\cdot, k) + O(k^{-\infty}).$$

*Furthermore the leading coefficient  $h_0$  satisfies*

$$D_t \iota^* h_0 = \iota^*(f_0 h_0), \quad h_0(t_0, x) = g_0(x), \quad \forall (t, x) \in I \times \Lambda$$

*where  $f_0$  and  $g_0$  are the leading coefficients of  $f(\cdot, k)$  and  $g(\cdot, k)$  respectively.*

*Proof.* Since the argument is very standard, we only sketch it. Using theorems 7.1 and 7.2, we construct by successive approximations a symbol  $h(\cdot, k)$  satisfying

$$\Pi_k((\nabla_{\partial_t}^{L^k \otimes \delta} + M_{f(\cdot, k)})(F^k h(\cdot, k))) = O(k^{-\infty})$$

and the initial condition

$$h(t_0, x, k) = g(x, k) + O(k^{-\infty}), \quad \forall x \in \Lambda$$

Then one proves that this approximate solution is equal to  $(\Psi_k)$  up to a  $O(k^{-\infty})$  term.  $\square$

We are now ready to prove theorem 6.3. Assume that 0 belongs to  $I$ . Denote by  $U_t : \mathcal{H}_{k,0} \rightarrow \mathcal{H}_{k,t}$  the parallel transport map for the connection

$$\Pi_k(\nabla_{\partial_t}^{L^k \otimes \delta} + M_{f(\cdot, k)})$$

Consider the symplectic manifold  $M' = M \times M^-$  with the prequantum bundle  $L_{M'} = L_M \boxtimes \bar{L}_M$ , the family of complex structures  $(j'_t = j_t \times -j_0)_{t \in I}$  and the family of half-form bundles  $(\delta'_t = \delta_t \otimes \bar{\delta}_0)_{t \in I}$ . We may apply all the previous constructions to these data. We denote them in the same way with a prime added. In particular,  $\mathcal{H}'_k \rightarrow I$  is the associated Hilbert space bundle. Its fiber at  $t$  is

$$\begin{aligned} \mathcal{H}'_{k,t} &= H^0(M_t \times \bar{M}_0, (L_t^k \boxtimes \bar{L}_0^k) \otimes (\delta_t \boxtimes \bar{\delta}_0)) \\ &\simeq \mathcal{H}_{k,t} \otimes \bar{\mathcal{H}}_{k,0} \\ &\simeq \text{End}(\mathcal{H}_{k,0}, \mathcal{H}_{k,t}) \end{aligned}$$

Denote by  $U$  the section of  $\mathcal{H}'_k$  whose value at  $t$  is  $U_t$ . It satisfies the following equation

$$\Pi'_k(\nabla_{\partial_t}^{(L')^k \otimes \delta'} + M_{f'(\cdot, k)})U = 0$$

where  $f'(\cdot, k)$  is the pull-back of  $f(\cdot, k)$  by the projection from  $I \times M^2$  onto the product of the first two factors. We apply theorem 7.4 to the section  $U$ . The Lagrangian submanifold  $\Lambda'$  is the diagonal of  $M^2$ , the section  $s'$  is given by  $s'(x, x) = u \otimes \bar{u}$ , for any  $u \in L_x$  of norm 1.  $U_0$  is the identity of  $\mathcal{H}_{k,0}$ . The fact that it satisfies the assumption of theorem 7.4 is the basic result of the semiclassical-analysis on compact Kähler manifolds. We deduced this result in [12] from the analysis of the Szegő kernel in [11]. It says that the Schwartz kernel of the identity of  $\mathcal{H}_{k,0}$  is equal to

$$\left(\frac{k}{2\pi}\right)^n F'_0 g'(\cdot, k) + O(k^{-\infty})$$

where  $F'_0$  is the restriction to  $M_0 \times \bar{M}_0$  of the section  $F'$  associated to  $(\Lambda', s')$ . The leading coefficient of  $g'(\cdot, k)$  satisfies  $g'(x, x, k) = u \otimes \bar{u} + O(k^{-1})$  for any  $u \in \delta_{0,x}$  of norm 1.

To conclude we only have to compute the leading coefficient of the solution. Identify the diagonal  $\Lambda'$  with  $M$ , so that the map  $\tilde{\varphi}'_{t,x}$  corresponding to  $\tilde{\varphi}_{t,x}$  sends  $(\delta_{t,x} \otimes \bar{\delta}_{0,x})^2$  to  $\wedge^{\text{top}} T_x^* M \otimes \mathbb{C}$ . Then a linear map  $\xi : \delta_{0,x} \rightarrow \delta_{t,x}$  is a half-form morphism if and only if

$$\tilde{\varphi}'_{t,x}((\xi(u) \otimes \bar{u})^2) = i^{n(n-2)} \omega_x^{\wedge n} / n!$$

for all  $u \in \delta_{0,x}$  of norm 1. This is the main reason at the origin of the definition of the half-form morphisms, cf. lemma 6.1 in [15] for a proof. The conclusion follows from the fact that  $\omega_x$  doesn't depend on  $t$ .

## References

- [1] Jorgen Ellegaard Andersen. The Witten invariant of finite order mapping tori I.
- [2] Jørgen Ellegaard Andersen. Asymptotic faithfulness of the quantum  $SU(n)$  representations of the mapping class groups. *Ann. of Math. (2)*, 163(1):347–368, 2006.
- [3] Jørgen Ellegaard Andersen. Mapping class groups do not have kazhdan's property (t). *ArXiv*, 0706.2184, 2007.
- [4] Jorgen Ellegaard Andersen, Niels Leth Gammelgaard, and Magnus Roed Lauridsen. Hitchin's connection in half-form quantization, 2007.
- [5] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.
- [6] Scott Axelrod, Steve Della Pietra, and Edward Witten. Geometric quantization of Chern-Simons gauge theory. *J. Differential Geom.*, 33(3):787–902, 1991.
- [7] Chris Beasley and Edward Witten. Non-abelian localization for Chern-Simons theory. *J. Differential Geom.*, 70(2):183–323, 2005.
- [8] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
- [9] Jean-Michel Bismut and François Labourie. Symplectic geometry and the Verlinde formulas. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 97–311. Int. Press, Boston, MA, 1999.



- [10] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.
- [11] L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegő. In *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, pages 123–164. Astérisque, No. 34–35. Soc. Math. France, Paris, 1976.
- [12] L. Charles. Berezin-Toeplitz operators, a semi-classical approach. *Comm. Math. Phys.*, 239(1-2):1–28, 2003.
- [13] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. *Comm. Partial Differential Equations*, 28(9-10):1527–1566, 2003.
- [14] L. Charles. Symbolic calculus for Toeplitz operators with half-form. *J. Symplectic Geom.*, 4(2):171–198, 2006.
- [15] L. Charles. Semi-classical properties of geometric quantization with metaplectic correction. *Comm. Math. Phys.*, 270(2):445–480, 2007.
- [16] L. Charles. Asymptotic properties of the quantum representations of the modular group. *ArXiv*, 1005.3447, 2010.
- [17] L. Charles. A note on the Chern-Simons bundle and the mapping class group. [http://people.math.jussieu.fr/~charles/Articles/Chern\\_Simons.pdf](http://people.math.jussieu.fr/~charles/Articles/Chern_Simons.pdf), 2010.
- [18] Laurent Charles. A lefschetz fixed point formula for symplectomorphisms. *Journal of Geometry and Physics*, 60(12):1890 – 1902, 2010.
- [19] Daniel S. Freed. Classical Chern-Simons theory. I. *Adv. Math.*, 113(2):237–303, 1995.
- [20] Michael H. Freedman, Kevin Walker, and Zhenghan Wang. Quantum  $SU(2)$  faithfully detects mapping class groups modulo center. *Geom. Topol.*, 6:523–539 (electronic), 2002.
- [21] Hajime Fujita. On the functoriality of the Chern-Simons line bundle and the determinant line bundle. *Commun. Contemp. Math.*, 8(6):715–735, 2006.
- [22] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990.

- [23] Lisa C. Jeffrey. Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation. *Comm. Math. Phys.*, 147(3):563–604, 1992.
- [24] Lisa C. Jeffrey. *On some aspects of Chern-Simons gauge theory*. PhD thesis, Oxford, 1992.
- [25] Julien Marché and Majid Narimannejad. Some asymptotics of topological quantum field theory via skein theory. *Duke Math. J.*, 141(3):573–587, 2008.
- [26] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *J. Differential Geom.*, 50(3):417–469, 1998.
- [27] E. Meinrenken and C. Woodward. Canonical bundles for Hamiltonian loop group manifolds. *Pacific J. Math.*, 198(2):477–487, 2001.
- [28] T. R. Ramadas, I. M. Singer, and J. Weitsman. Some comments on Chern-Simons gauge theory. *Comm. Math. Phys.*, 126(2):409–420, 1989.
- [29] S. Ramanan. The moduli spaces of vector bundles over an algebraic curve. *Math. Ann.*, 200:69–84, 1973.
- [30] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.
- [31] G. M. Tuynman. Quantization: towards a comparison between methods. *J. Math. Phys.*, 28(12):2829–2840, 1987.
- [32] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.